10. Optimum Design Problem Formulation

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Introduction to Problem Formulation

- Correct formulation of a problem takes roughly 50% of the total effort needed to solve
- Important to follow well-defined procedures for formulating design optimization problem
- Designing an engineering system is a complex process, which requires many assumptions, analyses by available methods, verification with experiments, etc.
- Economic considerations is another important aspect; cost-effective systems
- To complete the design of an engineering system, experts from different fields of engineering must usually cooperate: Multidisciplinary Design Optimization (MDO)
Introduction to Problem Formulation

- Optimum Design Problem Formulation is transcribing a verbal description of the problem into a well-defined mathematical statement.

- All systems are designed to perform within a given set of constraints
  - Limitation on resources, material failure, response of the system, member sizes, etc.
  - The constraints must be functions of the design variables.
  - If a design satisfies all constraints, we have a feasible (workable) system
  - If no design exist to satisfy the constraints, we call this as infeasible system

- A criterion is needed to judge whether or not a given design is better than another. This criterion is called the objective function or cost function.
  - An objective function must be a function of the design variables
Problem Formulation Procedure

• STEP 1: Identify and define design variables

• STEP 2: Identify the cost function and develop an expression for it in terms of design variables.

• STEP 3: Identify constraints and develop expressions for them in terms of design variables.
Example: Design of a Two-bar Structure

- Design a two-member bracket shown in the figure to support a force $W$ without structural failure.
- The force is applied at an angle $\theta$ which is between 0 and 90 degree, $h$ is the height and $s$ is the base width for the bracket.
- Since the brackets will be produced in large quantity, the design objective is to minimize its mass while also satisfying certain fabrication and space limitation.
STEP1: Identify Design Variables

- Design variables are parameters chosen to describe the design of a system
- At initial stage, designate as many design variables as possible
  - Which increases design flexibility
  - Later, we may eliminate (or fix) unimportant design variables using DOE, RSM, etc.
- All design variables should be independent of each other.
  - Example of circular tube \((d_i, d_o, \text{and } t)\)
- For the two-bar structure, \(h\) and \(s\) can be treated as design variables in the initial formation
- Other design variables will depend on the cross-sectional shape for members 1 and 2
STEP 2: Objective (Cost) Function

- There can be many feasible design for a system; We need some criterion to compare various designs.

- The criterion must be a scalar function whose numerical value can be obtained once a design is specified.

- Objective (or Cost function) is represented by $f$, or $f(x)$ to emphasize its dependence on design variable vector $x$.

- Maximization of $f(x)$ is converted as minimization of $-f(x)$.

- Objective functions may be:
  - minimize cost, maximize profit, minimize weight, minimize energy expenditure, maximize ride quality of vehicle, etc.

- Multiple objectives in cost function is called as multi-objective design optimization problem.
  - no general rule. We will visit this issue later.

- Proper formation of objective function is complex problem and needs experience and expertise of the system.
STEP2: Objective (Cost) Function

- In case of circular tube,
- Design variables are
  - $x_1 =$ height $h$ of the truss
  - $x_2 =$ span $s$ of the truss
  - $x_3 =$ outer diameter of member 1
  - $x_4 =$ inner diameter of member 1
  - $x_5 =$ outer diameter of member 2
  - $x_6 =$ inner diameter of member 2
- Objective function may be
  - Total mass of the truss
    $$\text{Mass} = \frac{\pi \rho}{8} (4x_1^2 + x_2^2)^{1/2} (x_3^2 + x_5^2 - x_4^2 - x_6^2)$$
    where
    $\rho$ is the density of the material
STEP 3: Design Constraints

- Feasible or infeasible design
- Implicit constraint is a constraint difficult to express as an explicit function of design variables
- Linear and Nonlinear constraints

\[
\begin{align*}
  x_1 + x_2 &\leq 1 \quad \text{(linear constraints)} \\
  x_1^2 - x_2^2 + 3x_1x_2 &\leq 0 \quad \text{(nonlinear constraints)}
\end{align*}
\]

- Equality and Inequality constraints

\[
\begin{align*}
  x_1 + x_2 &= 1 \quad \text{(equality constraints)} \\
  x_1 + x_2 &\leq 1 \quad \text{(inequality constraints)}
\end{align*}
\]
STEP 3: Design Constraints

- Principle of static equilibrium
  \[-F_1 \sin \alpha + F_2 \sin \alpha = W \cos \theta\]
  \[-F_1 \cos \alpha - F_2 \cos \alpha = W \sin \theta\]
  From the geometry,
  \[
  \sin \alpha = \frac{s}{2l} \\
  \cos \alpha = \frac{h}{l}
  \]
  \[
  F_1 = -0.5WL \left[ \frac{\sin \theta}{h} + \frac{2\cos \theta}{s} \right] \\
  F_2 = -0.5WL \left[ \frac{\sin \theta}{h} - \frac{2\cos \theta}{s} \right]
  \]
  where
  \[
  l = \sqrt{h^2 + (0.5s)^2}
  \]
STEP 3: Design Constraints

- Stress in a member is defined as force divided by cross-sectional area.
- To avoid over stressing of a member, the calculated stress must be less than or equal to material allowable stress (\( \sigma_a \), failure stress).

\[
\frac{2Wl}{\pi(x_3^2 - x_4^2)} \left[ \frac{\sin \theta}{h} + \frac{2 \cos \theta}{s} \right] - \frac{\sigma_a}{S} \leq 0
\]

\[
\frac{2Wl}{\pi(x_5^2 - x_6^2)} \left[ \frac{\sin \theta}{h} - \frac{2 \cos \theta}{s} \right] - \frac{\sigma_a}{S} \leq 0
\]

- Finally, constraints on design variables are written as

\[ x_{i_{-\text{low}}} \leq x_i \leq x_{i_{-\text{upper}}}; \quad i=1 \text{ to } 6 \]

which implies the bounds (minimum and maximum) of each design variable space.
Optimum Design Problem Formulation

- **Given**
  - Fixed variables, functions, assumptions

- **Find**
  - \( x \)  # design variables

- **Satisfy**
  - \( h(x) = 0 \)  # equality constraints
  - \( g(x) \leq 0 \)  # inequality constraints
  - \( x_{\text{low}} \leq x \leq x_{\text{upper}} \)  # bound on \( x \)

- **Minimize**
  - \( f(x) \)  # objective function

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**Verbal statement:** With **Given** fixed parameter \( \rho \) and \( S \), **Find** design variables \( x_1, x_2, x_3, x_4, x_5, x_6 \)
**Subject** to the constraints \( g_1, g_2 \), and bounds to **Minimize** the objective function.

**Given**

\[
\rho = 7.85, \quad S = 1.5
\]

**Find**

\[
x = (x_1, x_2, x_3, x_4, x_5, x_6)
\]

**Satisfy**

\[
\begin{align*}
g_1(x) &= \frac{2Wl}{\pi(x_3^2 - x_4^2)} \left[ \frac{\sin \theta}{h} + \frac{2\cos \theta}{s} \right] - \frac{\sigma_a}{S} \leq 0 \\
g_2(x) &= \frac{2Wl}{\pi(x_5^2 - x_6^2)} \left[ \frac{\sin \theta}{h} - \frac{2\cos \theta}{s} \right] - \frac{\sigma_a}{S} \leq 0
\end{align*}
\]

\[
x_{i_{\text{low}}} \leq x_i \leq x_{i_{\text{upper}}}; \quad i = 1 \text{ to } 6
\]

**Minimize**

\[
f(x) = \frac{\pi \rho}{8} (4x_1^2 + x_2^2)^{1/2} (x_3^2 + x_5^2 - x_4^2 - x_6^2)
\]
The cans will be produced in billions, so it is desirable to minimize the cost of manufacturing them. Since the cost can be related directly to the surface area of the sheet metal used, it is reasonable to minimize the sheet metal required to fabricate the can. Fabrication, handling aesthetic, and shipping considerations impose the following restrictions on the size of the can; (1) the diameter of the can should be no more than 8 cm; Also, it should not be less than 3.5 cm; (2) the height of the can should be no more than 18cm and no less than 8cm; and (3) the can is required to hold at least 400 ml of fluid.
Design of a Beer Can

- **STEP 1: Identify design variables**
  - Two design variables: \( D = \) diameter of the can (cm), \( H = \) height of the can (cm)

- **STEP 2: Identify objective function**
  - to minimize the total surface area of the sheet metal

  \[ \text{Surface Area} = \pi DH + \frac{\pi D^2}{2} \]

- **STEP 3: Identify constraints**
  - volume must be at least 400

  \[ \text{Volume} = \frac{\pi}{4} D^2 H \geq 400 \]
  - bounds on design variables

  \[ 3.5 \leq D \leq 8 \text{ and } 8 \leq H \leq 18 \]

\[ g(D, H) = 400 - \frac{\pi}{4} D^2 H \leq 0 \]

\[ 3.5 \leq D \leq 8 \text{ and } 8 \leq H \leq 18 \]

\[ f(D, H) = \pi DH + \frac{\pi D^2}{2} \]

Given
- Assumption of thin sheet metal

Find
- \( D = \) Diameter of can (cm)
- \( H = \) Height of can (cm)

Satisfy
- \( g(D, H) = 400 - \frac{\pi}{4} D^2 H \leq 0 \)
- \( 3.5 \leq D \leq 8 \text{ and } 8 \leq H \leq 18 \)

Minimize
- \( f(D, H) = \pi DH + \frac{\pi D^2}{2} \)
Example 2: Saw Mill Operation

A company owns two saw mills and two forests. Table below shows the capacity of each mill in logs/day and the distances between forests and mills. Each forest can yield up to 200 logs/day for the duration of the project and the cost to transport the logs is estimated at 15 cents/km/log. At least 300 logs are needed each day. Formulate the problem to minimize the cost of transportation of logs each day.

<table>
<thead>
<tr>
<th>Distance</th>
<th>Mill capacity/day</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Mill</td>
<td>Forest 1</td>
</tr>
<tr>
<td>A</td>
<td>24.0</td>
</tr>
<tr>
<td>B</td>
<td>17.2</td>
</tr>
</tbody>
</table>
Problem Formulation for Saw Mill Operation

Given

Cost to transport = 15 cents/km/log
Table of data for saw mill operation

Find

\[ x_1 = \text{number of logs shipped from Forest 1 to Mill A} \]
\[ x_2 = \text{number of logs shipped from Forest 2 to Mill A} \]
\[ x_3 = \text{number of logs shipped from Forest 1 to Mill B} \]
\[ x_4 = \text{number of logs shipped from Forest 2 to Mill B} \]

Satisfy

\[ x_1 + x_2 \leq 240 \quad \# \text{capacity constraint of Mill A} \]
\[ x_3 + x_4 \leq 300 \quad \# \text{capacity constraint of Mill B} \]
\[ x_1 + x_3 \leq 200 \quad \# \text{yield constraint of Forest 1} \]
\[ x_2 + x_4 \leq 200 \quad \# \text{yield constraint of Forest 2} \]
\[ x_1 + x_2 + x_3 + x_4 \geq 300 \quad \# \text{Constraints on the number of logs needed for a day} \]
\[ x_i \geq 0; \quad i = 1, 2, 3, 4 \quad \# \text{Bounds for reality} \]

Minimize

\[ f(x_1, x_2, x_3, x_4) = \text{cost} = 24(0.15)x_1 + 20.5(0.15)x_2 + 17.2(0.15)x_3 + 18(0.15)x_4 \]
Example 3: Design of Cabinet

A cabinet is assembled from components $C_1$, $C_2$ and $C_3$. Each cabinet requires eight $C_1$, five $C_2$ and fifteen $C_3$ components. Assembly of $C_1$ needs either five bolts or five rivets; $C_2$ six bolts or six rivets; and $C_3$ three bolts or three rivets. The cost of putting a bolt, including the cost of the bolt, is $0.70$ for $C_1$, $1.00$ for $C_2$ and $0.60$ for $C_3$. Similarly, riveting costs are $0.60$ for $C_1$, $0.80$ for $C_2$ and $1.00$ for $C_3$. A total of 100 cabinets must be assembled daily. Bolting and riveting capacities per day are 6000 and 8000, respectively. We wish to determine the number of components to be bolted and riveted to minimize the cost [after Siddall, 1972].
Problem Formulation

Given

Problem Definition

Find

\[ x_i = \text{number of C}_1 \text{ to be bolted} \]
\[ x_2 = \text{number of C}_1 \text{ to be riveted} \]
\[ x_3 = \text{number of C}_2 \text{ to be bolted} \]
\[ x_4 = \text{number of C}_2 \text{ to be riveted} \]
\[ x_5 = \text{number of C}_3 \text{ to be bolted} \]
\[ x_6 = \text{number of C}_3 \text{ to be bolted} \]

Satisfy

\[ x_1 + x_2 = 800 \]
\[ x_3 + x_4 = 500 \]
\[ x_5 + x_6 = 1500 \]
\[ 5x_1 + 6x_3 + 3x_5 \leq 6000 \]
\[ 5x_2 + 6x_4 + 3x_6 \leq 8000 \]
\[ x_i \geq 0; \quad i = 1 \text{ to } 6 \]

Minimize

\[ \text{Cost} = 0.70(5)x_1 + 0.60(5)x_2 + 1.00(6)x_3 
+ 0.80(6)x_4 + 0.60(3)x_5 + 1.00(3)x_6 \]
Example 4: Tubular Column Design

Straight columns as structural elements are used in many civil, mechanical, aerospace, agricultural and automotive structures. Many applications of columns can be observed in daily life, e.g. street light pole, traffic light post, flag pole, water tower support, highway sign post, power transmission poles, etc. It is important to design them as well as possible.

The problem is to design a minimum weight tubular column of length $l$ supporting a load $P$ without buckling or over stressing. The column is fixed at the base and free at the top. This type of structure is called a cantilever column. Buckling load for such a column is given as $\pi^2 EI/4l^2$ (buckling load for a column with other support conditions will be different from this formula [Crandall, Dahl and Lardner, 1978]). Here $I$ is the moment of inertia for the cross-section of the column and $E$ is the material property called modulus of elasticity (Young’s modulus). The material stress $\sigma$ for the column is defined as $P/A$, where $A$ is the cross-sectional area of the column material. The material allowable stress under axial load is $\sigma_a$, and material mass density is $\rho$ (mass per unit volume). Formulate the design problem.
Problem Formulation

**Given**

\[ P = 10 \text{ MN}, \quad E = 207 \text{Gpa}, \quad \rho = 7833 \text{kg/m}^3, \]
\[ l = 5.0, \quad \sigma_a = 248 \text{ Mpa} \]

**Find**

\( R, \ t \)

**Satisfy**

\[ g_1(R, t) = \frac{P}{2\pi R t} - \sigma_a \leq 0 \]
\[ = \frac{10(1.0E+06)}{2\pi R t} - 248(1.0E+06) \leq 0 \]
\[ \frac{10(1.0E+06)}{2\pi R t} - \frac{\pi^3 E R^3 t}{4l^2} \leq 0 \]
\[ = \frac{10(1.0E+06)}{4(5)(5)} - \frac{\pi^3 (207.0E+09) R^3 t}{4l^2} \leq 0 \]
\[ g_3(R, t) = -R \leq 0 \]
\[ g_4(R, t) = -t \leq 0 \]

**Minimize**

\[ f(R, t) = 2\pi \rho l R t \]
\[ = 2\pi (7833)(5) R t = (2.4608E+05) R t, \]
Example 5: Beam Design

- A beam of rectangular cross-section is subjected to a bending moment of $M$ (Nm) and maximum shear force of $V$ (N). The bending stress in the beam is calculated as $\sigma = 6M/bd^2$ (Pa) and the average shear stress is calculated as $\tau = 3V/2bd$ (Pa), where $b$ is the width and $d$ is the depth of the beam. The allowable stress in bending and shear are 10 MPa and 2 MPa respectively. It is desired to minimize cross-sectional area of the beam. For the numerical example, let $M = 40 \text{ kNm}$ and $V = 150 \text{ kN}$.
Problem Formulation

**Given**

\[ M = 40 \, kN \cdot m \, , \, V = 150kN \]

**Find**

\[ d = \text{depth of the beam (mm)} \]
\[ b = \text{width of the beam (mm)} \]

**Satisfy**

\[ g_1(b,d) = \frac{2.4 \times 10^8}{bd^2} - 10 \leq 0 \]
\[ g_2(b,d) = \frac{2.25 \times 10^5}{bd} - 2 \leq 0 \]
\[ g_3(b,d) = d - 2b \leq 0 \]

\[ b \geq 0 , \quad d \geq 0 \]

**Minimize**

\[ f(b,d) = bd \]
Example for Graphical Solution

A company manufactures two machines, A and B. Using available resources either 28 A or 14 B machines can be manufactured each day. The sales department can sell up to 14 A machines or 24 B machines. The shipping facility can handle no more than 16 machines per day. The company makes a profit of $400 on each A machine and $600 on each B machine. How many A and B machines should the company manufacture every day to maximize profit?
Graphical Solution

Problem Formulation

Given
Problem Description

Find

\[ x_1 = \text{number of A machines manufactured each day} \]
\[ x_2 = \text{number of B machines manufactured each day} \]

Satisfy

\[ x_1 + x_2 \leq 16 \quad \# \text{shipping and handling constraint} \]
\[ \frac{x_1}{28} + \frac{x_2}{14} \leq 1 \quad \# \text{manufacturing constraint} \]
\[ \frac{x_1}{14} + \frac{x_2}{24} \leq 1 \quad \# \text{sales constraint} \]

Minimize

\[ f(x_1, x_2) = -(400x_1 + 600x_2) \]

Linear Programming -> Simplex Method (Numerical Approach)
Design Problem with Multiple Solutions

Design Problem with Unbound Solutions

Subject to the constraints:

\[ 2x_1 + x_2 = 8 \]
\[ 2x_1 + 3x_2 = 12 \]

Optimum solution line B–C

\[ f = -1 \]
\[ f = -2 \]
\[ f = -3 \]

\[ 2x_2 = 0 \]
\[ 2x_1 + 3x_2 = 6 \]
\[ -2x_1 - 3x_2 = 6 \]

Cost function contours for the problem are pictured over the feasible region. All the points on and within the polygon ABCD and its complement—the set of points outside the feasible region in Fig. 2.7—are potential solutions to the problem. These points are continuous to the left and right, and the prices associated with the constraints on these points (reduction of profit) are constant. The general area for the use of the figures is ABCD, and the constraints on the cost functions and prices for the points in ABCD are shown in Fig. 2.7.
Graphical Solution for Tabular Column

Given

\[ P = 10 \text{ MN}, \ E = 207 \text{ Gpa}, \ \rho = 7833 \text{ kg/m}^3, \]
\[ l = 5.0, \ \sigma_a = 248 \text{ Mpa} \]

Find

\( R, \ t \)

Satisfy

\[ g_1(R, t) = \frac{P}{2\pi R t} - \frac{\sigma_a}{2} \leq 0 \]
\[ = \frac{10(1.0E+06)}{2\pi R t} - 248(1.0E+06) \leq 0 \]
\[ g_2(R, t) = P - \frac{\pi^3 ER^3 t}{4l^2} \leq 0 \]
\[ = 10(1.0E+06) - \frac{\pi^3(207.0E+09)R^3 t}{4(5)(5)} \leq 0 \]
\[ g_3(R, t) = -R \leq 0 \]
\[ g_4(R, t) = -t \leq 0 \]

Minimize

\[ f(R, t) = 2\pi \rho l R t \]
\[ = 2\pi (7833)(5)R t = (2.4608E+05)R t, \]
Graphical Solution for Beam Design

Given

\[ M = 40 \text{ kN} \cdot \text{m} \], \[ V = 150 \text{ kN} \]

Find

\( d = \) depth of the beam (mm)
\( b = \) width of the beam (mm)

Satisfy

\[ g_1(b,d) = \frac{2.4 \times 10^8}{bd^2} - 10 \leq 0 \]
\[ g_2(b,d) = \frac{2.25 \times 10^5}{bd} - 2 \leq 0 \]
\[ g_3(b,d) = d - 2b \leq 0 \]
\[ b \geq 0, \quad d \geq 0 \]

Minimize

\[ f(b,d) = bd \]